

On the Abel-Nörlund-Voronoi summability and instability of rational maps.

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Abstract

We investigate the connection between the instability of rational maps and summability methods applied to the spectrum of a critical point on the Julia set of a given rational map.

1 Motivation and main results

Let Rat_d be the space of all rational maps R of degree $d > 0$ defined on the Riemann sphere \mathbb{C} . The *postcritical set* is

$$P(R) = \overline{\bigcup_{c \in Crit(R)} \bigcup_{n \geq 0} R^n(c)}.$$

The *Julia set* J_R is the accumulation set of all repelling periodic cycles. A rational map R is called *hyperbolic* if the postcritical set does not intersect the Julia set.

The Fatou conjecture states that hyperbolic rational maps of degree d form an open and dense subset of Rat_d . According to Sullivan, a rational map $R \in Rat_d$ is called *structurally stable* if there exists a neighborhood U of R in Rat_d such that for every Q in U there is a quasiconformal conjugation between R and Q . A theorem of Mañé, Sad and Sullivan states that the set of structurally stable maps forms an open and dense subset of Rat_d (see [14]). Hence the Fatou conjecture can be restated in the following way.

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If $R \in \text{Rat}_d$ has a critical point c in the Julia set J_R then R is not structurally stable map or, equivalently, is an unstable map. Thus we are interested mostly in the study of orbits of critical points on the Julia set.

Let c be a critical point for R . We have the following trichotomy.

1. The sequence $(R^n)'(R(c))$ converges to 0.
2. There is a sequence n_i such that $(R^{n_i})'(R(c))$ converges to ∞ .
3. The number $\alpha = \liminf \frac{1}{|(R^n)'(R(c))|}$ is bounded and non-zero.

Given a generic point a , we call the sequence

$$\sigma(a) = \{\sigma_n(a)\} = \left\{ \frac{1}{(R^n)'(R(a))} \right\}_{n=0}^{\infty}$$

the spectrum of a . To avoid technicalities, we assume that ∞ is not in the orbit of any critical point, so the spectrum of all of them are well defined.

The *individual postcritical* set of a critical point c is

$$P_c = \overline{\bigcup_{n>0} R^n(c)}.$$

Let us comment the trichotomy above.

- Conjecturally the situation of (1) implies that the critical point c belongs to the Fatou set. For example suppose the sequence $(R^n)'(R(c))$ has bounded multiplicative oscillation, that is, if $\sup \left| \frac{(R^n)'(z)}{(R^{n+1})'(z)} \right|$ is bounded above, or equivalently, if $\liminf |R'(R^n(R(c)))| > 0$. Note that this implies that there are no critical points in the ω -limit set $\omega(c)$. Furthermore, assume $c \in J_R$ and that there are no parabolic cycles on $\omega(c)$, then Mañé's theorem (see [13]) implies that $\sigma(c)$ is absolutely summable which is a contradiction. In other words, if $\sigma(c)$ has bounded oscillation and R is structurally stable then c must belongs to the Fatou set.
- In the situation (3), assume $\sigma(c) \in \ell_\infty$ then the sequence has bounded multiplicative oscillation and, as in the previous item, by Mañé's theorem the map R is not structurally stable.

The following observations are the main motivation of the thrichotomy above and the present work.

Proposition 1.1. *Let R be a stable rational map and c be a critical point on the Julia set $J(R)$ with bounded individual postcritical set P_c . Consider the partial sums $S_n = \sum_{j=0}^n \frac{1}{(R^j)'(R(c))}$. Then for all n we have*

$$|S_n| \leq C |\sigma_n(c)|,$$

for some constant C .

The proof is contained in Lemma 5 in [12] (see also A. Avila [1]). The quadratic polynomial case was also noted by G. Levin in [9]. The proof, however is elementary and we include it here for the sake of completeness.

Proof. The proof is by contradiction. Given a stable map R , consider the family of rational maps $R_t = R + t$ for $|t| < 1$. According to Mañé, Sad and Sullivan ([14]) there exists a holomorphic family f_t of quasiconformal automorphisms of \mathbb{C} such that, for ϵ small enough, given t with $|t| < \epsilon$ we have

$$R_t = f_t \circ R \circ f_t^{-1}.$$

Let $F(z) = \frac{\partial f_t}{\partial t}|_{t=0}(z)$ be the variation of f_t . Then F is a continuous function on \mathbb{C} . By a straightforward computation of the variation of R_t we obtain

$$F(R(z)) - R'(z)F(z) = 1.$$

Hence we get $F(R(c)) = 1$ and by iteration of the equation above we arrive to

$$F(R^n(R(c))) = (R^n)'(R(c))S_n.$$

We end up with $|S_n| \leq \sup_{z \in P_c} |F(z)\sigma_n(c)|$, and since F is bounded on compacts subsets of the plane we are done. \square

We get the following elementary corollaries.

Corollary 1.2. *Let R be a rational map and c be a critical point on the Julia set $J(R)$ with bounded P_c . Either one of the following conditions imply that the map R is unstable.*

- *In the situation of (2), the sequence $\sigma_{n_i}(c)$ converges to 0 and satisfies $\limsup |S_{n_i}| > 0$.*
- *In the situation of (3), the sequence $\sigma_{n_i}(c)$ converges to a non-zero finite limit α and $\limsup |S_{n_i}| = \infty$.*

In the next corollary instead of the spectrum $\sigma(c)$ of a critical point c , we consider the sequence of finite barycenters

$$b_n = \frac{1}{n}S_{n-1}.$$

We say that $\sigma(c)$ *barycentrically converges* when $\{b_n\}$ converges.

Corollary 1.3. *Let R be a rational map and c be a critical point such that $\sigma_n(c) = o(n)$ and the individual postcritical set P_c is bounded. Assume that $\sigma(c)$ does not converge to 0 barycentrically. Then R is an unstable map.*

Proof. By contradiction, if R is a stable rational map, by assumption and Proposition 1.1 there exists a constant C such that

$$|b_n| \leq \frac{C}{n}|\sigma_{n-1}(c)|.$$

As the right hand side of the latter converges to 0, the spectrum $\sigma(c)$ barycentrically converges also to 0, which is a contradiction. \square

Let us note that if a sequence converges then the sequence of finite barycenters converges to the same limit. The reciprocal fails to be true: consider for instance an oscillating sequence between two terms. Moreover, many unbounded sequences are barycentrically convergent.

On the other hand, by results due to A. Avila [1], G. Levin [9] and P. Makienko [12], if the series $\sum_{n=0}^{\infty} \sigma_n(c)$ is absolutely convergent then the map R is unstable whenever the individual postcritical set P_c satisfies further additional conditions (which always hold, for example, in the case of polynomials). Let us note that a good behavior of the spectrum (such as convergence or, even better, absolute summability) forces extra conditions on the individual postcritical sets; compare with Corollary 1.2 and the work of Avila, Levin and Makienko cited above.

Among other facts, in [12] it was shown that if $c \in J_R$ is non-recurrent and the series $\sum_{n=0}^{\infty} \sigma_n(c)$ is absolutely convergent, then R is unstable. However, it is yet unclear whether a structurally stable map may not have a critical point $c \in J_R$ with $\sigma(c) \in \ell_{\infty}$.

In this article we answer this question positively, with additional topological assumptions on the individual postcritical set P_c , on the following theorem.

Theorem 1.4. *Let c be a critical point in J_R with bounded P_c and $\sigma(c) \in \ell_{\infty}$. Assume that c and P_c are separated by the Fatou set, then we have the following.*

1. *The map R is unstable whenever P_c has measure 0.*
2. *If $\sigma(c)$ is convergent then R is unstable.*

Theorem 1.4 is a consequence of more general statements which are the core of this article.

Other sources of inspiration for this work are the following.

In the case of unimodal polynomials G. Levin, F. Przytycki and W. Shen ([10]), showed that the radius of convergence of the power series with coefficients of the spectrum is always 1 whenever c belongs to the Julia set. In what follows, we will understand by the radius of convergence of a sequence as the radius of convergence of the corresponding power series.

Also for unimodal polynomials, H. Bruin and S. van Strien [4] (see also J. Rivera-Letelier [16]) proved that if $\sigma(c) \in \ell_1$, then the Lebesgue measure of $J(R)$ is 0. For rational maps of higher degree the situation is more complicated (see Avila [1], Levin [9], Makienko [12]).

Finally, an example given by A. Avila and M. Lyubich [2] of a quadratic polynomial with Julia set of positive measure shows that it is possible to have a critical point $c \in J(R)$ with $\sigma(c) \notin \ell_1$. Indeed our work is devoted to investigate sequences which have non-zero radius of convergence but are not absolutely convergent.

1.1 Abel measures

First we discuss some weaker conditions on the behavior of $\sigma(c)$, for $c \in J(R)$, that include many instances of (1), (2) and (3) above. Instead of the sequence

of barycenters we consider a more general situation, namely a sequence of Abel averages.

Definition 1.5. Let a_n be a sequence with radius of convergence at least 1. For every complex number λ with $|\lambda| < 1$ the value $P_\lambda = (1 - \lambda) \sum a_n \lambda^n$ is called an *Abel average* of a_n . If for λ real the limit $\lim_{\lambda \rightarrow 1} P_\lambda$ exists, then $\{a_n\}$ is called *Abel convergent*.

For example, if $\{a_n\}$ barycentrically converges then $\{a_n\}$ is Abel convergent to the same limit. In the setting of Abel convergence Theorem 1.4 becomes the following proposition.

Proposition 1.6. Assume we have a critical point $c \in J(R)$ with $\sigma_n(c) = o(n)$ and P_c bounded. If $\sigma(c)$ is not Abel convergent to 0 then R is unstable.

If the series $\sum a_n$ is absolutely convergent then the complex Abel averages uniformly converge to 0 for every sequence of complex numbers λ_i with $|\lambda_i| \leq 1$ and converging to 1.

For every λ with $|\lambda| < 1$, the Abel average P_λ defines a linear functional on ℓ_∞ whose norm is $\|P_\lambda\| = \frac{|1-\lambda|}{1-|\lambda|}$. In particular, the family of functionals P_λ is unbounded when λ approaches to 1 tangentially, even on \mathcal{C}_0 . This implies that, in general, the tangential limit of P_λ is different from 0 even in the case of conditional convergence of $\sum a_n$. If a tangential limit of P_λ is not a radial limit, then the cluster set of P_λ is a continuum subset of the Riemann sphere. This might happen even when the sequence $\{a_n\}$ belongs to $\mathcal{C}_0 \setminus \ell_1$, here \mathcal{C}_0 and ℓ_1 are the spaces of complex sequences converging to 0 and absolutely summable complex sequences, respectively.

If the series $\sum a_n$ is conditionally convergent then the Abel limit of $\{a_n\}$ is always 0. However, in general, the tangential limits are different from 0, specially for sequences with unbounded multiplicative oscillation. The discussion above leads to the following definition.

For a point $z \in \mathbb{C}$, consider the orbit $R^n(z)$. Given a complex Abel average P_λ we associate the measure

$$\nu_\lambda(z) = (1 - \lambda) \sum a_n \lambda^n \delta_{R^n(z)}$$

where δ_y is the Dirac measure based on y .

Definition 1.7. A finite complex valued measure ν is an *Abel measure for the sequence a_n with respect to the point z* whenever there exists a complex sequence λ_i converging to 1 and a sequence of complex numbers r_i so that ν is the $*$ -weak limit of the measures $r_i \nu_{\lambda_i}$.

In other words, the complex projective classes of ν_{λ_i} converge to the complex projective class of ν as functionals on continuous functions. When $\{a_n\} = \sigma(a)$ and $z = R(a)$ we will refer to the associated Abel measure as the *Abel measure with respect to the point a* . In this event, the support of ν belongs to the closure of the orbit $R^n(z)$.

If the sequence a_n belongs to ℓ_∞ and if $\frac{|1-\lambda_i|}{1-|\lambda_i|} = \alpha > 0$, then the family of measures ν_{λ_i} is uniformly bounded, and therefore has accumulation points. So, whenever $\lim_j \int d\nu_{i_j} = \lim_j P_{\lambda_{i_j}} \neq 0$ for a suitable subsequence i_j , then every $*$ -weak limit of this sequence is a non-zero measure.

If there exists a non-zero Abel measure, then often it is not unique. In particular, this is the generic case for sequences $\{a_n\} \in \ell_\infty \setminus \mathcal{C}_0$, where \mathcal{C}_0 is the space of converging sequences.

We call a finite complex valued measure μ a *Mergelyan type measure*, or an *M-measure* for short, when its Cauchy transform $f_\mu(z) = \int_{\mathbb{C}} \frac{d\mu(t)}{t-z}$ is not identically 0 outside the support of μ .

For $K \subset \mathbb{C}$ compact, consider $C(K)$ the space of continuous functions on K and $Rat(K)$ the restriction on K of rational functions with poles outside of K . If $Rat(K)$ is dense $C(K)$ with respect to uniform convergence, then any complex finite measure supported on K is an *M-measure*. When either $m(K) = 0$ or $\inf\{diam(W) : W \text{ component of } \mathbb{C} \setminus K\} > 0$, by classical results, we have that $\overline{Rat(K)} = C(K)$. For a more complete treatment of the theory see for example the book of T. Gamelin [6].

The following lemma gives a simple criterion to determine whether a measure ν is an *M-measure*.

Lemma 1.8. *Let K be a compact set in \mathbb{C} and $\nu \neq 0$ be a finite measure with $\text{supp } \nu \subset \bigcup \partial W_i$, where the W_i are the connected components of $\mathbb{C} \setminus K$. Then ν is an *M-measure*.*

Proof. We use a classical argument from [6], see also the article by Makienko [12]. By Theorem 10.4 in [6] if ν is a complex finite measure then the measurable locally integrable function f given by $f(t) = \int \frac{d\nu}{z-t}$ vanishes almost everywhere if and only if $\nu = 0$.

We proceed by contradiction. We will assume that ν is not an *M-measure* and show that $f(t)$ is 0 almost everywhere on the boundary ∂W of any component W of $\mathbb{C} \setminus K$. Let W_∞ be the connected component which contains ∞ and W be any other component of $\mathbb{C} \setminus K$, possibly empty. For the compact set $C_W = \mathbb{C} \setminus W_\infty \cup W$ Theorem 10.4 on [6] establishes that the closure of $Rat(C_W)$ consists of continuous functions on C_W which are holomorphic on the interior of C_W . Let $E \subset \partial W_\infty \cup \partial W$ be a measurable subset of positive Lebesgue measure. We claim that

$$\int_E f(t) |dt|^2 = 0.$$

Indeed, in this case the function

$$F_E(z) = \int \chi_E \frac{1}{t-z} |dt|^2$$

is continuous on \mathbb{C} and holomorphic outside of the closure of E . By the above

discussion and Fubini's theorem we have

$$\begin{aligned} 0 &= \int F_E(z) d\nu(z) = \int d\nu \int_E \frac{|dt|^2}{t-z} \\ &= - \int_E |dt|^2 \int \frac{d\nu(z)}{(z-t)} = - \int_E f(t) |dt|^2, \end{aligned}$$

which settles our claim and gives the contradiction. \square

As a consequence of the previous lemma, if R is a map that has a completely invariant Fatou component, then every finite measure supported on J_R is an M -measure. In particular, every finite measure supported on the Julia set of a polynomial is an M -measure.

Now we are ready to formulate our first theorem.

Theorem 1.9 (Theorem A). *Assume that c is a critical point in J_R such that its individual postcritical set P_c does not contain all the fixed points of R . Let ν_0 be an Abel measure with respect to c . Then R is unstable whenever ν_0 is Mergelyan.*

Let us note that Theorem 1.9 imposes restrictions only over one individual postcritical set. This does not exclude the possibility that the whole postcritical set coincides with the Riemann sphere. So our arguments allow us to work in this difficult case also.

As shown in Corollary 1.2 and Proposition 1.6, in the worst case scenario (such as when the spectrum is not barcentrically convergent to 0) the condition $c \in J_R$ implies the instability of R almost without restrictions on P_c . On the other side, it was shown in [12] for the best case scenario (the absolute convergence of $\sigma(c)$) that there do exist non-zero Abel measures. However, we believe that any non-absolutely summable sequence with unbounded multiplicative oscillation and barcentrically convergent to 0 always posses a non-zero Abel measure. Between these extreme situations we have the following theorem which complements Theorem 1.9 under an extra topological assumption on an individual postcritical set P_c .

Theorem 1.10 (Theorem B). *Assume that c is a critical point in J_R such that $\sigma(c)$ has radius of convergence at least 1. Let $|\lambda_i| < 1$ be a sequence of complex numbers converging to 1 and such that ν_{λ_i} converges $*$ -weakly to 0. If c and P_c are separated by the Fatou set $F(R)$, then R is unstable.*

In general, Theorem 1.9 corresponds to all cases in the thrichotomy whereas Theorem 1.10 mostly corresponds to case (2).

1.2 Voronoi measures

Now we generalize the approach above to a subclass of sequences mostly associated to case (1) of the thrichotomy. We will use the so called Nörlund summability method which was published firstly by G. Voronoi in 1902 (see [17]).

Fix a sequence of non-negative real numbers $q_n \geq 0$ subject to $q_0 > 0$ and $\lim_n \frac{q_n}{Q_n} = 0$, where $Q_n = q_0 + \dots + q_n$ are the partial sums. For a sequence of complex numbers $\{x_n\}$, the values

$$t_n = \frac{q_n x_0 + q_{n-1} x_1 + \dots + q_0 x_n}{Q_n}$$

are called the *Nörlund averages* with respect to the sequence $\{q_n\}$. If

$$\limsup \sqrt[n]{|t_n|} \leq 1,$$

then we say that $\{x_n\}$ is *Nörlund regular* (or *N-regular* for short).

In fact, this method defines a linear operator $N : \Theta \rightarrow \Theta$ where Θ is the linear space of all complex sequences, here N can be identified with the infinite matrix $(\alpha_{m,n})$, where $\alpha_{m,n} = \frac{q_{m-n}}{Q_m}$ for $n \leq m$ and $\alpha_{m,n} = 0$ for $n > m$. It is known (see for example [3]) that N defines a continuous linear endomorphism of ℓ_∞ . Moreover we have $N(\mathcal{C}) \subset \mathcal{C}$, and further $\lim_n x_n = \lim_n t_n$. Here \mathcal{C} is the space of convergent sequences.

Next lemma is not difficult to prove and appears as Lemma 3.3.10 in [3].

Lemma 1.11. *Let N be the Nörlund matrix $(\frac{q_{m-n}}{Q_m})$, and $\{x_n\}$ be a Nörlund regular sequence. Then the following properties hold.*

1. *The series $q(\lambda) = \sum q_n \lambda^n$ and $Q(\lambda) = \sum Q_n \lambda^n$ converge for $|\lambda| < 1$.*
2. *The series $x(\lambda) = \sum x_n \lambda^n$ converges in a neighborhood of 0.*
3. *The convolution series*

$$[N(x_n)](\lambda) = \sum_n \left(\sum_{i=0}^n q_i x_{n-i} \right) \lambda^n = \sum_n t_n Q_n \lambda^n$$

converges for $|\lambda| < 1$.

If $\{x_n\}$ is *N-regular*, from the relation $[N(x_n)](\lambda) = x(\lambda)q(\lambda)$ and $q_0 \neq 0$ we conclude two facts. First, the radius of convergence of $\{x_n\}$ is non-zero and $x(\lambda)$ can be continued to a meromorphic function $X(\lambda)$ on the unit disk. Second, as $q(\lambda)$ has non negative Taylor coefficients, there are no zeros on the interval $[0, 1)$, and so the poles of $X(\lambda)$ lay outside $[0, 1)$. Reciprocally, if the sequence $\{x_n\}$ has non-zero radius of convergence and $x(\lambda) = \sum x_n \lambda^n$ can be extended to a meromorphic function $X(\lambda) = \frac{\psi(\lambda)}{\phi(\lambda)}$ where ϕ and ψ are holomorphic and $\phi(\lambda) > 0$ holds on the unit interval $[0, 1)$ with non negative sequence of its Taylor coefficients, then the Taylor coefficients of ϕ defines a Nörlund operator N such that $\{x_n\}$ is a *N-regular* sequence.

So the Nörlund method is a generalization of the Abel method for some sequences with *non-zero radius of convergence*. However, let us note that geometrically divergent sequences are not Nörlund regular. In particular, for a

structurally stable R , if $\sigma(c)$ is Nörlund regular then $c \in J_R$. We conclude that Nörlund regular sequences define a subclass of case (1) of the thrichotomy.

Now we follow the construction done for Abel convergence applied to the Nörlund method. Given a point $z \in \mathbb{C}$, a Nörlund matrix $N = (\frac{q_{m-n}}{Q_m})$ and a sequence $\{x_n\}$ such that $N(\{|x_i|\}) = \{t_n\}$ has radius of convergence 1, we associate the measure

$$\nu_\lambda = (1 - \lambda) \sum_n T_n \lambda^n,$$

where $T_n = q_n x_0 \delta_z + q_{n-1} x_1 \delta_{R(z)} + \dots + q_0 x_n \delta_{R^n(z)}$. Then by part (3) of Lemma 1.11, the ν_λ form an analytic family of finite measures over the open unit disk.

Definition 1.12. We call a finite complex valued measure ν a *Voronoi measure with respect to N and z* if there exists a sequence of complex numbers $\lambda_i \rightarrow 1$ (with $|\lambda_i| < 1$) such that the complex projective classes $[\nu_{\lambda_i}]$ converge to the complex projective class of $[\nu]$. If $\{x_n\} = \sigma(c)$ and $z = R(c)$ for a suitable critical point $c \in J_R$ then we call ν the *Voronoi measure associated to the point c* .

We have the following two theorems.

Theorem 1.13. *Suppose that for a critical point c in the Julia set $J(R)$ the sequence $\{|\sigma_n(c)|\}$ is N -regular for a Nörlund matrix N . If a non-zero Voronoi measure ν associated to c is also an M -measure, then R is unstable.*

Next theorem is complementary to the theorem above.

Theorem 1.14. *Let λ_i be a sequence of complex numbers with $|\lambda_i| < 1$ and $\lambda_i \rightarrow 1$ for which ν_{λ_i} is $*$ -weak convergent to 0. Then R is an unstable map whenever c and P_c are separated by the Fatou set.*

The results on this paper are based mostly upon ideas of the second author.

2 Some background in dynamics and Poincaré series

To prove our theorems, we need first some preparation. Most of the background material can be found in [12] (see also [11]).

A rational map R defines a complex pushforward map on $L_1(\mathbb{C})$, with respect to the Lebesgue measure m . This contracting endomorphism is called the *complex Ruelle-Perron-Frobenius*, or the *Ruelle operator* for short. The Ruelle operator is explicitly given by the formula

$$R_*(\phi)(z) = \sum_{y \in R^{-1}(z)} \frac{\phi(y)}{R'(y)^2} = \sum_i \phi(\zeta_i(z)) (\zeta'_i(z))^2,$$

where ζ_i is any local complete system of branches of the inverse of R . The Beltrami operator $Bel : L_\infty(\mathbb{C}) \rightarrow L_\infty(\mathbb{C})$ given by

$$Bel(\mu) = \mu(R) \frac{\overline{R'}}{R'}$$

is dual to the Ruelle operator acting on $L_1(\mathbb{C})$. The fixed point space $Fix(B)$ of the Beltrami operator is called the *space of invariant Beltrami differentials*.

Every element $\mu \in L_\infty(\mathbb{C})$ defines a continuous function on \mathbb{C} via

$$F_\mu(a) = a(a-1) \int_{\mathbb{C}} \frac{\mu(z)}{z(z-1)(z-a)} |dz|^2,$$

which is called the *normalized potential for μ* . By convenience we write $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}$ so that we get $F_\mu(z) = \int \gamma_a(z) \mu(z) |dz|^2$.

The following statement appears as Lemma 5 and 6 in [12].

Lemma 2.1. *Let R be a structurally stable rational map. Then for every critical value v_i there exists an invariant Beltrami differential μ_i such that $F_{\mu_i}(v_j) = \delta_{i,j}$, the delta Kroenecker function.*

Now we give the formal relations of Poincaré-Ruelle series.

Definition 2.2. The *Poincaré-Ruelle series* are

•

$$B_a(z) = \sum_{n \geq 0} (R_*)^n(\gamma_a(z)),$$

•

$$A_a(z) = \sum_{n=0} \frac{1}{(R^n)'(a)} \gamma_{R^n(a)}(z).$$

The lemma below gives a formal relation between both Poincaré-Ruelle series. The proof can be found on [12] and further details are contained in [11].

Lemma 2.3. *Let R be a rational map, fixing 0, 1 and ∞ , with simple critical points c_i and set $v_i = R(c_i)$. Let a be a value so that $\bigcup_n \{R^n(a)\}$ does not contain critical points. Then we have the following formal relation between the above series*

$$B_a(z) = A_a(z) + \sum_i \frac{1}{R''(c_i)} A_a(c_i) \otimes B_{v_i}(z) \quad (*)$$

where \otimes is the formal Cauchy product. Hence we have

$$\begin{aligned} (R_*)^n(\gamma_a(z)) &= \frac{1}{(R^n)'(a)} \gamma_{R^n(a)}(z) + \sum_i \frac{1}{R''(c_i)} \left[\frac{\gamma_{R^{n-1}(a)}(c_i)}{(R^{n-1})'(a)} \gamma_{v_i}(z) + \right. \\ &\quad \left. + \frac{\gamma_{R^{n-2}(a)}(c_i)}{(R^{n-2})'(a)} R_*(\gamma_{v_i}(z)) + \cdots + \gamma_a(c_i) (R_*)^{n-1}(\gamma_{v_i}(z)) \right]. \end{aligned}$$

To the formal series $A_a(z)$ and $B_a(z)$, involved in Equation (*), we associate formal Abel series parameterized by the unit disk as follows. For $|\lambda| < 1$, write

$$A_a(z, \lambda) = \sum_n \frac{\lambda^n}{(R^n)'(R(a))} \gamma_{R^n(a)}(z)$$

and

$$B_a(z, \lambda) = \sum_n \lambda^n (R_*)^n(\gamma_a(z)).$$

Then we have the following lemma.

Lemma 2.4. *Let R be a structurally stable rational map and $c \in J(R)$ be a critical point whose spectrum $\sigma(c)$ has radius of convergence $r > 0$. Take $v = R(c)$. Then for any complex number λ with $|\lambda| < r$ we have the following.*

1. *The series $A_v(z, \lambda)$ is absolutely convergent almost everywhere with respect to z and is an integrable function holomorphic off P_c .*
2. *Let \tilde{c} be a critical point. The numerical series $A_v(\tilde{c}, \lambda)$ is absolutely convergent.*
3. *The series $B_a(z, \lambda)$ is absolutely convergent almost everywhere and is an integrable function for every $|\lambda| < 1$ and every $a \in \mathbb{C}$.*

Furthermore, each of the series above define a holomorphic function with respect to λ for $|\lambda| < r$.

Proof. By assumption the series $\sum \lambda^n \sigma_n(c)$ is absolutely convergent for $|\lambda| < r$ and defines a holomorphic function with respect to λ in the disk $|\lambda| < r$.

Part (1). As there exists a constant C such that $\int |\gamma_a(z)| |dz|^2 \leq C|a| \ln |a|$ (see for example the books by Gardiner-Lakič [7] and Krushkal [8]) we get

$$\begin{aligned} \int_{\mathbb{C}} |A_v(z, \lambda)| |dz|^2 &\leq \sum |\lambda^n \sigma_n(c)| \int_{\mathbb{C}} |\gamma_{R^n(v)}(z)| |dz|^2 \\ &\leq C \sum |\lambda^n \sigma_n(c)| R^n(v) \ln |R^n(v)|. \end{aligned}$$

The last expression is absolutely convergent. Moreover, by the mean value theorem the series $A_v(z, \lambda)$ converges uniformly on compact sets outside $P_c \cup \{0, 1, \infty\}$.

Part (2). If $\tilde{c} \notin P_c$ then by part (1) we are done. Assume that $\tilde{c} \in P_c$. Given $\epsilon > 0$ let U_ϵ be the ϵ neighborhood of \tilde{c} . Let n_i be such that $R^{n_i}(v) \in U_\epsilon$. As in the arguments in Part(1), it is enough to estimate the expression

$$\left| \sum_i \lambda^{n_i} \sigma_{n_i}(c) \gamma_{R^{n_i}(v)}(\tilde{c}) \right|.$$

Note that for $z \in U_\epsilon$, we have $R'(z) = (z - \tilde{c})R''(\tilde{c}) + O(|z - \tilde{c}|^2)$. Hence

$$\left| \frac{1}{R^{n_i}(v) - \tilde{c}} \right| \leq \left| \frac{R''(\tilde{c}) + O(|R^{n_i}(v) - \tilde{c}|)}{R'(R^{n_i}(v))} \right| \leq M \left| \frac{1}{R'(R^{n_i}(v))} \right|$$

and

$$|\gamma_{R^{n_i}(v)}(\tilde{c})| \leq M \left| \frac{1}{R'(R^{n_i}(v))} \cdot \frac{R^{n_i}(v)(R^{n_i}(v) - 1)}{\tilde{c}(\tilde{c} - 1)} \right| \leq M_1 \left| \frac{1}{R'(R^{n_i}(v))} \right|,$$

where M and M_1 are suitable constants depending on ϵ and \tilde{c} . As a result of the previous computation we obtain

$$\begin{aligned} \left| \sum_i \lambda^{n_i} \sigma_{n_i}(c) \gamma_{R^{n_i}(v)}(\tilde{c}) \right| &\leq M_1 \sum_i \left| \frac{\lambda^{n_i} \sigma_{n_i}(c)}{R'(R^{n_i}(v))} \right| \\ &\leq \frac{1}{|\lambda|} M_1 \sum |\lambda^{n_i+1} \sigma_{n_i+1}(c)| < \infty, \end{aligned}$$

for $0 < |\lambda| < 1$.

Part (3). Since $\|R_*(f)\|_{L_1} \leq \|f\|_{L_1}$ holds for any $f \in L_1(\mathbb{C})$, then for any $|\lambda| < 1$, the series $B_a(z, \lambda)$ is an integrable function and so converges absolutely almost everywhere for every $a \in \mathbb{C}$. \square

We have the following immediate corollary.

Corollary 2.5. *If the spectrum $\sigma(c)$ of a critical point c has a radius of convergence $r > 0$ then we can rewrite Equation (*) as*

$$B_{R(c)}(z, \lambda) = A_{R(c)}(z, \lambda) + \lambda \sum_i \frac{1}{R''(c_i)} A_{R(c)}(c_i, \lambda) \cdot B_{v_i}(z, \lambda), \quad (**)$$

for every $|\lambda| < r$ and almost every $z \in \mathbb{C}$.

Proof. This comes from the previous lemma and the Cauchy product theorem. \square

3 Proofs of the theorems.

3.1 Abel case

Now we are ready to proof the theorems. We start with an observation.

Lemma 3.1. *For $|\lambda_i| < 1$ assume that $r_i \nu_{\lambda_i}$ *-weakly converges to ν_0 , where ν_0 is an Abel measure for a critical point c for which $\sigma(c)$ has radius of convergence at least 1. If R is structurally stable then the sequence $r_i A_v(\tilde{c}, \lambda_i)$ is bounded and convergent for any critical point \tilde{c} .*

Proof. For $\tilde{v} = R(\tilde{c})$ take an invariant Beltrami differential $\tilde{\mu}$ as in Lemma 2.1. Integrating Equation (**) in Corollary 2.5 with respect to $\tilde{\mu}$ give

$$\begin{aligned} \int_{\mathbb{C}} \tilde{\mu}(z) B_v(z, \lambda) |dz|^2 &= \int_{\mathbb{C}} \tilde{\mu}(z) A_v(z, \lambda) |dz|^2 + \\ &+ \lambda \sum_i \frac{1}{R''(c_i)} A_v(c_i, \lambda) \cdot \int_{\mathbb{C}} \tilde{\mu}(z) B_{v_i}(z, \lambda) |dz|^2. \end{aligned}$$

As $\tilde{\mu}$ is invariant we get

$$\int_{\mathbb{C}} \tilde{\mu}(z) B_{v_i}(z, \lambda) |dz|^2 = \frac{1}{1 - \lambda} F_{\tilde{\mu}}(v).$$

By the choice of $\tilde{\mu}$, after multiplying by $r_i(1 - \lambda_i)$ on both sides, we obtain

$$r_i(1 - \lambda_i) \int \tilde{\mu}(z) A_v(z, \lambda_i) |dz|^2 = r_i \left[F_{\tilde{\mu}}(v) - \frac{\lambda_i}{R''(c)} A_v(c, \lambda_i) \right]. \quad (1)$$

On the other hand, we have $(1 - \lambda_i) A_v(z, \lambda_i) = \int \gamma_a(z) d\nu_{\lambda_i}(a)$. Hence by Fubini theorem we get

$$r_i(1 - \lambda_i) \int \tilde{\mu}(z) A_v(z, \lambda_i) |dz|^2 = \int d\nu_{\lambda_i}(a) \int \gamma_a(z) \tilde{\mu}(z) |dz|^2.$$

We take limits

$$\lim_{i \rightarrow \infty} \int F_{\tilde{\mu}}(a) r_i d\nu_{\lambda_i}(a) = \int F_{\tilde{\mu}}(a) d\nu_0(a)$$

and return to (1) to finish the proof. \square

Proposition 3.2. *If R is structurally stable then $\phi(z) = \int \gamma_a(z) d\nu_0(a)$, where ν_0 is as in the Lemma 3.1, is integrable and satisfies $R_*(\phi(z)) = \phi(z)$.*

Proof. First, again by Fubini's theorem we get

$$\begin{aligned} \int |\phi(z)| |dz|^2 &\leq \int |d\nu_0(a)| \int |\gamma_a(z)| |dz|^2 \\ &\leq M \int |a| |\ln |a|| d\nu_0(a) < \infty. \end{aligned}$$

Therefore ϕ is integrable. Applying $r_i(1 - \lambda_i)[Id - \lambda_i R_*]$ to Equation (**) in Corollary 2.5 with $\lambda = \lambda_i$, after using the resolvent equation $(Id - \lambda R_*) \circ (\sum \lambda^i (R_*)^i) = Id$, we have

$$\begin{aligned} r_i(1 - \lambda_i) \gamma_{v_1}(z) &= r_i(1 - \lambda_i) [Id - \lambda_i \cdot R_*] (A_{v_1}(z, \lambda_i) + \\ &\quad + (1 - \lambda_i) \lambda_i \sum_j \frac{1}{R''(c_j)} A_{v_1}(c_j, \lambda_i) \cdot \gamma_{v_j}(z)). \end{aligned}$$

Now

$$\begin{aligned} r_i(1 - \lambda_i) [Id - \lambda_i R_*] A_{v_1}(z, \lambda_i) &= [Id - R_*] [r_i(1 - \lambda_i) A_{v_1}(z, \lambda_i)] + \\ &\quad + R_* [r_i(1 - \lambda_i)^2 A_{v_1}(z, \lambda_i)]. \end{aligned}$$

Since $\phi(z) = \lim_{i \rightarrow \infty} r_i(1 - \lambda_i) A_{v_1}(z, \lambda_i)$ holds almost everywhere and R_* is continuous with respect to almost everywhere convergence, we apply Lemma 3.1 and we are done. \square

Proof of Theorem A. By assumption the measure ν_0 of the Lemma 3.1 is an M -measure. Then $\nu_0 \neq 0$ and direct computations show $\phi(z) \neq 0$ on $\mathbb{C} \setminus P_{c_1}$. By Corollary 12 in [12] and Lemma 3.16 of [15] (see also [5]), R is a flexible Lattés map which is a contradiction with structural stability. \square

Now we give two criteria for a sequence to have a non-zero Abel measure. The first is elementary and, roughly speaking, says that if the arguments of the sequences a_i are close enough to 0 then this sequence has a non-zero Abel measure with respect to every rational map R and $z \in \bar{\mathbb{C}}$. More precisely, we have the following lemma.

Lemma 3.3. *Let $\{a_n\}$ be a complex sequence with radius of convergence at least 1. Assume that there exist $\alpha < 1$ and a complex sequence λ_i converging to 1 with $|\lambda_i| < 1$ such that*

$$|\lambda_i^n a_n - |\lambda_i^n a_n|| \leq \alpha |\lambda_i^n a_n|.$$

Then for every rational map R and every point $z \in \mathbb{C}$, there exists a non-zero Abel measure with respect to $\{a_n\}$ and z .

Proof. Fix R and $z \in \mathbb{C}$. Then every $*$ -weak limit of the family of probability measures

$$w_\lambda = \frac{\sum |\lambda|^n |a_n| \delta_{R^n(R(z))}}{\sum |\lambda|^n |a_n|}$$

is a probability measure. Write

$$u_\lambda = \frac{\sum \lambda^n a_n \delta_{R^n(R(z))}}{\sum |\lambda|^n |a_n|},$$

so that u_λ is a family of complex measures absolutely continuous with respect to w_λ . For the sequence λ_i assume that u_{λ_i} converges $*$ -weakly to 0. Define $X = \bigcup R^n(z)$ and let 1_X be the characteristic function on X . Notice that the supports of w_λ and u_λ belong to X . Now the inequality

$$\begin{aligned} 1 &= \lim_{\lambda_i \rightarrow 1} \left| \int 1_X du_{\lambda_i} - \int 1_X dw_{\lambda_i} \right| \\ &= \lim_{\lambda_i \rightarrow 1} \frac{|\sum_n \lambda_i^n a_n - \sum_n |\lambda_i^n a_n||}{\sum_n |\lambda_i^n a_n|} \\ &\leq \lim_{\lambda_i \rightarrow 1} \frac{\sum_n |\lambda_i^n a_n - |\lambda_i^n a_n||}{\sum_n |\lambda_i^n a_n|} \leq \alpha < 1 \end{aligned}$$

establishes a contradiction. \square

The second criteria is connected with the L_1 norm of the function $A_z(\lambda)$ on $\mathbb{C} \setminus P_c$. Indeed, we prove a more general statement.

Lemma 3.4. *Let $K \subset \mathbb{C}$ be compact and let ν_i be a bounded sequence of complex valued measures on K . If we assume*

$$\limsup_i \int_{\mathbb{C}} \left| \int_{\mathbb{C}} \gamma_a(z) d\nu_i(a) \right| |dz|^2 > 0,$$

then there exists a $$ -weak accumulation point ν_0 which is not null. The reciprocal is also true.*

Proof. According to a well known result on quasiconformal theory (see for example F. Gardiner and N. Lakič [7] or S. L. Krushkal [8]) the operator $T : L_{\infty}(\bar{\mathbb{C}}) \rightarrow C(K)$ given by

$$T(\mu)(a) = \int_{\mathbb{C}} \gamma_a(z) \mu(z) |dz|^2,$$

which maps μ to $F_{\mu}|_K$, is continuous and compact. The same is true for the dual operator $T^* : M(K) \rightarrow (L_{\infty}(\mathbb{C}))^*$ given by

$$T^*(m)(z) = \int_K \gamma_a(z) dm(a)$$

is continuous and compact too.

By Fubini's theorem $\text{rank}(T^*) \subset L_1(\mathbb{C})$ and each $T^*(m)(z)$ is holomorphic outside of $K \cup \{0, 1, \infty\}$. Hence $T^* : M(K) \rightarrow L_1(\mathbb{C})$ is a compact operator. Now, by assumption, passing to a subsequence, we have that

$$f_i(z) = \int_K \gamma_a(z) d\nu_i(a) = T^*(\nu_i)(z)$$

converges in norm to a non-zero f in $L_1(\mathbb{C})$.

Note that $\partial_{\bar{z}} f_i = \nu_i$ in the sense of distributions. If ν_i converges $*$ -weakly to 0, then, by continuity, the integrability of f and an application of Weyl's lemma we have $f = 0$ which is a contradiction, so ν_i cannot converge $*$ -weakly to 0.

Reciprocally, if a measure $\nu_0 \neq 0$ is a $*$ -weak limit of ν_i then the measurable function $T^*(\nu_0) \neq 0$ on \mathbb{C} and $T^*(\nu_i)$ converges to $T^*(\nu_0)$ by norm. \square

To prove Theorem B we need some additional preparation. Recall that a positive Lebesgue measurable subset W of $\bar{\mathbb{C}}$ is called *wandering* if $R^{-n}(W)$ forms a family of pairwise almost disjoint sets with respect to the Lebesgue measure. The union $D(R)$ of all wandering sets is called the *dissipative set*, its complement $\bar{\mathbb{C}} \setminus D(R)$ is the *conservative set*. A Fatou component U belongs to $D(R)$ precisely when U is not a periodic rotation component.

Theorem 3.5. *Let R be a structurally stable map. If R satisfies the conditions of Theorem B. Then,*

$$\lim_{i \rightarrow \infty} \frac{1}{1 - \lambda_i} \int \gamma_z(a) d\nu_{\lambda_i}(a) = 0$$

almost everywhere on $D(R)$. Moreover, on the Fatou set the limit above is uniform on compact sets outside of the postcritical set $P(R)$.

Proof. First let us show that for any $\phi \in L_1(\mathbb{C})$ the series $\sum R_*^n(\phi)$ is finite and converges absolutely on $D(R)$. It is enough to show that $\sum |R_*^n(\phi)|$ is integrable on any wandering set W . Direct computations show

$$\int_W |R_*^n(\phi(z))| |dz|^2 \leq \int_{R^{-n}(W)} |\phi(z)| |dz|^2,$$

so adding leads us to

$$\int_W \sum |R_*^n(\phi(z))| \leq \sum \int_{R^{-n}(W)} |\phi(z)| |dz|^2 \leq \int_{\mathbb{C}} |\phi(z)| |dz|^2.$$

Hence $R_*^n(\phi(z))$ converges to 0 almost everywhere on $D(R)$.

Second, if $z_0 \in F(R) \setminus P(R)$, then there exist a disk $D_0 \subset D(R)$ centered at z_0 . Now suppose that ϕ is holomorphic on $\mathbb{C} \setminus P(R)$. Since we have

$$\int_{D_0} |R_*^n(\phi(z))| |dz|^2 \leq \int_{\mathbb{C}} |\phi(z)| |dz|^2,$$

by the mean value theorem $R_*^n(\phi)$ forms a normal family of holomorphic functions on D_0 . By the discussion above R_*^n and $\sum R_*^n(\phi(z))$ converge to their respective limits uniformly on compact subsets of D_0 . Then by the Abel theorem we get

$$\lim_{\lambda_i \rightarrow 1} \sum \lambda_i^n R_*^n(\phi(z)) = \sum R_*^n(\phi(z))$$

almost everywhere on $D(R)$ and uniformly on compact subsets of $F(R) \setminus P(R)$.

Now we claim that if \tilde{c} is a critical point then

$$\lim_{\lambda_i \rightarrow 1} A_{R(c)}(\tilde{c}, \lambda) = \begin{cases} 0 & \text{if } \tilde{c} \neq c \\ R''(c) & \text{if } \tilde{c} = c. \end{cases}$$

In fact, Lemma 2.1 provides invariant Beltrami differential $\tilde{\mu}$ with

$$\int_{\mathbb{C}} \tilde{\mu} \gamma_{R(\tilde{c})}(z) |dz|^2 = \begin{cases} 1 & \text{if } \tilde{c} \neq c \\ 0 & \text{if } \tilde{c} = c. \end{cases}$$

Integrating Equation (**) of Corollary 2.5 against $\tilde{\mu}$ as in Lemma 3.1, taking limits with respect to λ_i and applying the assumption that all $*$ -weak limits of ν_i are 0, we get our claim.

To finish the proof, we take $\lambda_i \rightarrow 1$ in Equation (**) of Corollary 2.5. In order to get

$$\lim_{i \rightarrow \infty} A_{R(c)}(z, \lambda_i) = \lim_{i \rightarrow \infty} \frac{1}{1 - \lambda_i} \int_{\mathbb{C}} \gamma_z(a) d\nu_{\lambda_i}(a) = 0$$

almost everywhere on the dissipative set $D(R)$ and uniformly on compact subsets of $F(R) \setminus P(R)$. \square

Proof of Theorem B. By contradiction, assume that R is structurally stable. Then as in the claim in the proof of Theorem 3.5 we get

$$\lim_{\lambda_i \rightarrow 1} A_{R(c)}(c, \lambda_i) = R''(c) \neq 0.$$

On the other hand, $A_{R(c)}(z, \lambda_i)$ converges to 0 uniformly on compact subsets of $F(R) \setminus P(R)$. By assumption we can select a Jordan curve $g \subset F(R) \setminus P(R)$ separating c and P_c . By Cauchy's theorem we have

$$\lim_{\lambda_i \rightarrow 1} A_{R(c)}(c, \lambda_i) = \int_g \frac{A_{R(c)}(z, \lambda_i)}{z - c} dz = 0,$$

a contradiction. \square

Proof of Theorem 1.4. If R were structurally stable then the measures ν_λ would form a uniformly bounded family for $0 \leq \lambda < 1$. However, by Theorem B, every $*$ -weak limit of ν_λ for $\lambda \rightarrow 1$ is a non-zero Abel measure. Let $\tau \neq 0$ be such an Abel measure as above. Since P_c has measure 0 then τ is an M -measure and which is in conflict with Theorem A, so the part (1) is done.

Part (2). Let $\tau \neq 0$ be an Abel measure which is not an M -measure. Then

$$f(z) = \int_{\mathbb{C}} \gamma_a(z) d\tau(a)$$

is a non-zero integrable function on \mathbb{C} supported on P_c satisfying $R_*(f) = f$. By Lemma 11 in [12] there exists an invariant Beltrami differential μ with $\mu(z) = \frac{|f(z)|}{f(z)}$ almost everywhere on $\text{supp}(f)$. Computations give

$$\begin{aligned} 0 &\neq \int_{\mathbb{C}} |f| |dz|^2 = \int_{\mathbb{C}} \mu(z) f(z) |dz|^2 \\ &= \lim_{\lambda_i \rightarrow 1} (1 - \lambda_i) \int_{\mathbb{C}} \mu(z) A_v(z, \lambda_i) |dz|^2 \\ &= \lim_{\lambda_i \rightarrow 1} (1 - \lambda_i) \left[\left(\frac{1}{R''(c)} A_v(c, \lambda_i) - 1 \right) \int_{\mathbb{C}} \mu(z) B_v(z, \lambda_i) |dz|^2 + \right. \\ &\quad \left. + \sum_{\tilde{c} \in \text{Crit}(R) \setminus \{c\}} \frac{1}{R''(\tilde{c})} A_v(\tilde{c}, \lambda_i) \int_{\mathbb{C}} \mu(z) B_{\tilde{v}}(z, \lambda_i) |dz|^2 \right]. \end{aligned}$$

By invariance this reduces to

$$0 \neq F_\mu(v) \lim_{\lambda_i \rightarrow 1} \left[\frac{1}{R''(c)} A_v(c, \lambda_i) - 1 \right] + \sum_{\tilde{c} \in \text{Crit}(R) \setminus \{c\}} \frac{F_\mu(\tilde{v})}{R''(\tilde{c})} \lim_{\lambda_i \rightarrow 1} A_v(\tilde{c}, \lambda_i). \quad (2)$$

Now, if R is stable and since $\sigma(c)$ is convergent then, by Corollary 1.2 the sequence $\sigma(c)$ converges to 0. Let $s_n(\tilde{c})$ be the partial sums of the formal series

$A_v(\bar{c})$, here \bar{c} is any critical point. Then by assumptions, Lemma 2.1 and the formula after Lemma 2.4 together yield

$$\lim_{i \rightarrow \infty} s_n(\bar{c}) = \begin{cases} 0 & \text{if } \bar{c} \neq c \\ R''(c) & \text{if } \bar{c} = c. \end{cases}$$

Abel's theorem then gives

$$\lim_{\lambda \rightarrow 1} A_v(\bar{c}, \lambda) = (1 - \lambda) \sum_n s_n(\bar{c}) \lambda^n = \begin{cases} 0 & \text{if } \bar{c} \neq c \\ R''(c) & \text{if } \bar{c} = c. \end{cases}$$

replacing these values in (2) gives the desired contradiction. \square

With small modifications the theorems above can be extended to the case of entire or meromorphic functions with finitely many critical and asymptotic values.

3.2 The Nörlund-Voronoi case

The key idea of the proof of Theorem 1.13 resembles the context of Theorem A and Theorem 3.5.

Proof of Theorem 1.13. As $|\sigma_n(c)|$ is Nörlund regular with respect to the matrix $N = \{\frac{q_{n-m}}{Q_n}\}$, it has radius of convergence $r > 0$. Hence

$$e_\lambda = (1 - \lambda) \sum \lambda^n \sigma_n(c) \delta_{R^n(c)}$$

is a finite measure for $|\lambda| < r$. By part (3) of Lemma 1.11 we have

$$q(\lambda) \cdot e_\lambda = \nu_\lambda.$$

Thus e_λ can be extended to the open unit disk as a meromorphic family of measures, which are holomorphic on a neighborhood of $[0, 1)$. Let $E_\lambda = \frac{\nu_\lambda}{q(\lambda)}$ be the induced extension. In this way the function

$$A_v(z, \lambda) = \frac{1}{1 - \lambda} \int_{\mathbb{C}} \gamma_a(z) d e_\lambda(a)$$

extends to

$$\begin{aligned} E_v(z, \lambda) &= \frac{1}{q(\lambda)(1 - \lambda)} \int_{\mathbb{C}} \gamma_a(z) d \nu_\lambda(a) \\ &= \frac{1}{(1 - \lambda)} \int_{\mathbb{C}} \gamma_a(z) d E_\lambda(a). \end{aligned}$$

In other words, for z outside $P_c \cup \{0, 1, \infty\}$ the sequence $\{\sigma_n(c) \gamma_{R^n(v)}(z)\}$ is Nörlund regular with respect to N .

Like in the proofs of Lemma 2.1 and Corollary 2.5, we extend $A_v(\tilde{c}, \lambda)$ to a meromorphic function $E_v(\tilde{c}, \lambda)$ for every critical point \tilde{c} so that

$$B_v(z, \lambda) = E_v(z, \lambda) + \lambda \sum_{c_i \in \text{Crit}(R)} \frac{1}{R''(c_i)} E_v(c_i, \lambda) B_{v_i}(z, \lambda). \quad (***)$$

holds on a neighborhood of $[0, 1)$.

Under the assumptions, the arguments of Proposition 3.2 apply. \square

Proof of Theorem 1.14. We proceed as in Theorem B, but we only need to apply Lemma 2.1 and Theorem 3.5 to Equation (***). If R is structurally stable we get a contradiction to the assumptions proceeding as in Theorem B. \square

To conclude, let us review the main ideas of this article. We considered various averaging methods to produce finite non-zero measures, such as Abel and Voronoi measures.

Following these ideas, a general averaging mechanism can be hinted to establish the Fatou conjecture for a larger class of sequences. Recall that Θ is the space of all complex sequences. A matrix M is *regular* if when restricted to \mathcal{C} it defines a continuous operator such that if $\{t_n\} = M\{a_n\}$ then $\lim a_n = \lim t_n$. A basic fact here is Agnew's Theorem which states that given $x, y \in \Theta$ if either

1. $x \in \ell_\infty \setminus \mathcal{C}$ and $y \in \ell_\infty$ or
2. $x \in \Theta \setminus \ell_\infty$ and $y \in \Theta$

then there is regular matrix M with $Mx = y$ (see Theorem 2.6.4 in [3]).

Now, given an infinite matrix M such that

- i) If $c \in J_R$ then M is such that the measures $\rho_k = \sum_{n=0}^k \delta_{R^n(v)} \sigma_n(c)$ are transformed by M onto measures t_k with $\sup_k \|t_k\| < \infty$ and there is a non-zero α which is a $*$ -weak accumulation point of t_k . The Agnew's theorem guarantees the existence of such a matrix for non convergent complex sequences.
- ii) Let us recall that ℓ_1 acts on Θ by convolutions. Let M be a matrix satisfying the previous condition. If M commutes with the action of ℓ_1 by convolutions, then using the formal Equation (*) we can show the instability of the corresponding rational map.

Let us note that the algebra of linear operators on Θ commuting with the action by convolutions of ℓ_1 is non-separable. But, not every matrix given by Agnew Theorem commutes with convolution on ℓ_1 . On the other hand, most of the summation methods discussed in the book of Boos [3] satisfy both conditions. In this article we used only two of them, namely Abel and Nörlund-Voronoi methods.

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